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The Schrödinger equation for the $x^2 + \lambda x^2/(1 + gx^2)$ interaction

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Abstract. Certain exact odd and even wavefunctions for the potential $x^2 + \lambda x^2/(1 + gx^2)$, $x \in (-\infty, \infty)$ are obtained. The coefficients of the wavefunctions and expressions for the energies are obtained using the symbolic manipulation code REDUCE. The chief use of these results is to provide a check for the effectiveness of numerical algorithms. The results supplement those of Flessas for the same potential on $x \in (0, \infty)$.

1. Introduction

The eigenvalue problem

$$y'' + [\varepsilon - x^2 - \lambda x^2/(1 + gx^2)]y = 0 \quad (1.1)$$

(where $g \geq 0$) and its three-dimensional analogue have attracted some attention in recent years. Flessas (1981, 1982) obtained some exact solutions when ε and λ are specifically related to g . He considered the problem on the interval $(0, \infty)$. In this paper we study the potential on the interval $(-\infty, \infty)$ and provide a definite procedure for the calculation of odd and even eigenstates. In the process we explain an observation made in Flessas (1981) to the effect that in the limit $g \rightarrow 0$ the n th wavefunction for (1.1) apparently reduces to the $(n - 2)$ th wavefunction for the harmonic oscillator.

The eigenvalue problem (1.1) occurs in two areas of physical interest. In quantum field theory (1.1) becomes a one-dimensional Schrödinger equation associated with a zero-dimensional field theory which has applications in elementary particle physics. In laser physics (1.1) is obtained by the reduction of the Fokker-Planck equation of a single-mode laser under appropriate conditions (see Kaushal 1979 and references therein). The eigenvalues and eigenfunctions have been calculated numerically by Mitra (1978) and Bessis and Bessis (1980). The convergence of the fixed-point expansion was proven by Znojil (1984). Since the exact solutions obtained require a definite relationship between λ and g , in general for a given permissible potential one may find just one eigenvalue. Thus in practice numerical methods must be used. Apart from the intrinsic interest of solving an eigenvalue problem exactly even if under restricted conditions, our solutions provide an opportunity to check numerical results by comparing them with precise results. This is in line with previous analyses we have made of polynomial anharmonic oscillators (Leach 1984, 1985).

2. Structure of the exact wavefunction

In our previous study (Leach 1985) of one-dimensional anharmonic oscillators in which the anharmonicity was expressed as a polynomial in x , the wavefunction was found to be of the form

$$y = f(x) \exp(-h(x)) \quad (2.1)$$

where f and h were polynomials in x with the highest power in h being even and having a positive coefficient. This is not the case with (1.1) since, if f is of degree M and h of degree $2N+2$, (1.1) (multiplied by $(1+gx^2)$) has a single term of degree $M+4N+4$ in x and so its coefficient is zero. By a recursive argument it follows that $N=0$. This is confirmed by examining the asymptotic behaviour of (1.1) which gives $y \sim \exp(-\frac{1}{2}x^2)$. Since the potential in (1.1) is even, the wavefunction is either even or odd. Normally one would make the ansatz that the wavefunction has the form $P_n(x) \exp(-\frac{1}{2}x^2)$ where P_n is an even or odd polynomial. However, when one examines the wavefunction for small values of N , the factor $(1+gx^2)$ is found. This leads us to revise the conventional ansatz and assume that the wavefunction has the form

$$y(x) = \sum_{n=0}^N c_n x^{2n+\delta} (1+gx^2) \exp(-\frac{1}{2}x^2) \quad (2.2)$$

where δ is zero or one depending upon whether we are looking for even or odd wavefunctions. From general Sturm-Liouville theory, for real x , $y(x)$ has only simple zeros and so $c_0 \neq 0$.

Substituting (2.2) into (1.1) and multiplying by $1+gx^2$, we separate coefficients of different powers of x . Since the coefficient of the highest power must be zero, the value of λ is given by

$$\lambda = -g[4(N+1)+2\delta+1-\varepsilon] \quad (2.3)$$

i.e. λ depends upon the eigenvalue. Thus we expect to obtain only one eigenstate per potential unless it happens that for different choices of N we obtain the same value of λ . We return to this point below. With λ given by (2.3), we obtain the following recurrence relation for the coefficients c_n :

$$c_{n+1} = -[(2n+\delta+1)(2n+\delta+2)]^{-1} \{ [(2n+\delta+1)(2n+\delta+2)g - 2(2n+\delta) + \varepsilon - 1]c_n + 4g(N+1-n)c_{n-1} \} \quad n=0, N-1 \quad (2.4)$$

and we take c_{-1} to be zero.

The eigenvalues are obtained from the solutions of the equation

$$[(2N+\delta+1)(2N+\delta+2)g - 2(2N+\delta) + \varepsilon - 1]c_N + 4gc_{N-1} = 0 \quad (2.5)$$

which follows from (2.4) with $c_{N+1} = 0$. This gives an equation in ε and g which is to be solved for ε to find the eigenvalues.

The actual calculation of the coefficients is readily undertaken by a simple REDUCE procedure. They rapidly become very complex as N increases. However, REDUCE handled the calculations without any difficulty, the time of computation being of the order of 850 ms on an IBM 370K to calculate all the c for $N=0, 6$.

3. More than one eigenvalue per potential

In § 2 we mentioned the possibility that this method could provide more than one eigenvalue per potential. It is evident that for given values of N and δ this will not be the case as the different values of ϵ give different λ and so different potentials. If we wish to obtain the same λ for a choice of N_1, δ_1 and N_2, δ_2 then ϵ_2 must be related to ϵ_1 by

$$\epsilon_2 = \epsilon_1 + 4(N_2 - N_1) + 2(\delta_2 - \delta_1). \quad (3.1)$$

We can give closed form expressions for ϵ in terms of g only for low values of N . If we restrict our attention to $N = 0, 1$, we find that the same value of λ does occur for specific values of g . Setting $N_1 = 0, N_2 = 1, \delta_1 = 0$ and $\delta_2 = 1$, (3.1) becomes

$$\epsilon_2 = \epsilon_1 + 6. \quad (3.2)$$

In terms of g we calculate that

$$\epsilon_1 = 1 - 2g \quad \epsilon_2 = 5 - 3g \pm (49g^2 - 4g + 4)^{1/2}. \quad (3.3)$$

Substituting these expressions into (3.2) we obtain

$$\pm (49g^2 - 4g + 4)^{1/2} = g + 2. \quad (3.4)$$

As $g \geq 0$ we must take the positive root in (3.2) and find that a non-trivial solution (i.e. $g \neq 0$) occurs if $g = \frac{1}{6}$. Hence we find that the potential

$$V(x) = x^2 - x^2/(18 + 3x^2) \quad (3.5)$$

has the eigenvalues

$$\epsilon_0 = \frac{2}{3} \quad \epsilon_3 = 6\frac{2}{3} \quad (3.6)$$

with the corresponding wavefunctions (unnormalised)

$$\begin{aligned} \psi_0 &= (1 + \frac{1}{6}x^2) \exp(-\frac{1}{2}x^2) \\ \psi_3 &= x(1 - \frac{7}{3}x^2)(1 + \frac{1}{6}x^2) \exp(-\frac{1}{2}x^2) \end{aligned} \quad (3.7)$$

i.e. we have the ground state and the third excited state. (We have replaced ϵ_1 and ϵ_2 by ϵ_0 and ϵ_3 in the normal practice of using the subscript to indicate the energy level. *A priori* we do not know which states are going to be obtained.) Likewise for the choice $N_1 = 0, N_2 = 1, \delta_1 = 1$ and $\delta_2 = 0$ we find that $g = \frac{2}{3}$ and for

$$V(x) = x^2 - 8x^2/(3 + 2x^2) \quad (3.8)$$

$$\epsilon_1 = -1 \quad \epsilon_2 = 1 \quad (3.9)$$

$$\begin{aligned} \psi_1 &= x(1 + \frac{2}{3}x^2) \exp(-\frac{1}{2}x^2) \\ \psi_2 &= (1 - \frac{2}{3}x^2)(1 + \frac{2}{3}x^2) \exp(-\frac{1}{2}x^2) \end{aligned} \quad (3.10)$$

i.e. the first and second excited states. One could continue on searching for such instances, but the going becomes tougher. For $N_1 = 0, \delta_1 = 0, N_2 = 2$ and $\delta_2 = 0$, there is no positive g which gives equal λ . For $N_1 = 0, \delta_1 = 1, N_2 = 2$ and $\delta_2 = 0$, the value of g is 1.2918... and that of λ , -15.1797... A calculation of the coefficients c_i and

c_2 for the $N_2=2$ shows that the wavefunction has only two zeros and so we have the first and second excited states with

$$\varepsilon_1 = -4.7508 \dots \quad \varepsilon_2 = 1.2491 \dots \quad (3.11)$$

$$\psi_1(x) = x(1 + 1.2918x^2) \exp(-\frac{1}{2}x^2) \quad (3.12)$$

$$\psi_2(x) = (1 - 1.4664x^3 - 3.1278x^4)(1 + 1.2918x^2) \exp(-\frac{1}{2}x^2).$$

4. Dependence of eigenvalues on g

In the simple example discussed in § 3 for which it was possible to write ε in terms of g in closed form (see (3.3)) it is evident that the asymptotic dependence of ε on g is linear. In general, it is not possible to obtain such closed form expressions and, indeed, the equations relating ε and g become very complex even for small N . However, it is a simple matter to obtain numerical values for ε given a value of g . If one plots ε against g for various values of N , it is evident that (i) ε is asymptotically linear in g with the asymptotic behaviour becoming well established for $g < 1$ in the case $N = 5$ and (ii) the constant multiplier becomes less negative the higher the state becomes. The dominance of the linear dependence of ε on g is easy to see from a table of differences. In the case of $N = 5$ and $\delta = 0$ with g incrementing by steps of 1, in the range $19 \leq g \leq 31$, the first difference was around -132 and the second difference of the order of one thousandth of that. From (2.3) it is clear that asymptotically λ will be quadratic in g .

5. Conclusion

We have seen that analytic solutions of the eigenvalue problem (1.1) can be found in a systematic way when ε and λ are related to g . Generally there is only one eigenvalue found for a given λ since λ depends upon ε . However, by appropriate choices of N and g we can cover a wide variety of potentials of this class. This would be useful to check the accuracy of approximate procedures and in particular to check the actual amount of computation required to give the desired accuracy. This may help to reduce the cost of numerical computations.

In the introduction we mentioned an observation by Flessas (1981) that in the limit $g \rightarrow 0$ the n th wavefunction reduces to the $(n-2)$ th wavefunction for the harmonic oscillator. In fact this is not the case. As $g \rightarrow 0$, the n th wavefunction tends smoothly to the n th wavefunction of the harmonic oscillator. The reason is found in § 2 in which it was observed that each wavefunction contained a non-zero factor $(1+gx^2)$. This means that the n th wavefunction has a polynomial of degree $n+2$ multiplying the exponential term. As g becomes zero the degree of the polynomial becomes n , but the number of real zeros remains unchanged.

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